Axisymmetric Potential Flow in Ducts

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A method and numerical solution is described for the investigation of axisymmetric potential flow in ducts which contain a central body. The contours of the body and the duct are represented by continuous distributions of ring vortices. The boundary conditions on infinity are satisfied by source discs, closing the inlet and outlet sections of the duct. This model is applicable also to multiconnected axisymmetric regions. Numerical results for bodies in a tube of constant radius are presented and compared with other solutions. The stability and accuracy of the computations is observed for various geometry and blockages.

I. Introduction

THE investigation of flows about bodies in channels and ducts is of interest in a number of applications like turbomachines and for the purpose of more accurate determination of wall corrections for model tests in wind and cavitation tunnels and ship-model towing tanks.

Several methods of determining the axisymmetric irrotational flow about bodies of revolution in a duct of constant radius have been proposed.[1-4] In such cases (Ref. 1), the contribution of the duct can be introduced in the kernel of the integral equation for the strength of the vortex distribution. This yields to a complication of the kernel with an integral sometimes requiring special treatment.

The use of a vortex model for the body and the duct in Ref. 2 is more appropriate for the generalization of the problem for an arbitrarily shaped duct, but it is not given by the authors because they found some difficulties in the problem for a tube of constant radius.

II. Formulation of Problem

As shown in Fig. 1, the body and the duct are represented by vortex distribution along its contours. The boundary condition of zero flow within the body and out of the duct leads to the Dirichlet problem

$$\frac{1}{2}\gamma(s)=u(s)\frac{\mathrm{d}x}{\mathrm{d}s}+v(s)\frac{\mathrm{d}r}{\mathrm{d}s} \tag{1}$$

where (x,r) are cylindrical coordinates, s is contour coordinate, u and v are local velocity in axial and radial direction. A well known relation on the surface is

$$W(s)=\gamma(s) \tag{2}$$

Here $W(s)$ is the velocity at the exterior side of the body surface or in the interior of the duct wall surface.

For the purpose of closing the vortex system and satisfying the boundary condition on infinity in front of and behind the body, the constant cylindrical parts of the flow are represented by semi-infinite vortex cylinders, identical in the outer flow with source discs.[5,6] The velocity components can be then expressed in the form

$$u(s)=-\frac{1}{2\pi}\int\frac{u_B^*}{r(\sigma)}\gamma_B(\sigma)\mathrm{d}\sigma+\frac{1}{2\pi}\int\frac{u_D^*}{r(\sigma)}\gamma_D(\sigma)\mathrm{d}\sigma+u_{D_1}+u_{D_2} \tag{3}$$

with a similar form for $v(s)$, where the subscript B denotes body, D denotes duct, and u^*, v^* are the nondimensional velocity components[6] induced by a vortex ring of unit radius.

Introducing Eq. (3) and the corresponding expressions for $v(s)$, Eq. (1) becomes

$$g^B(x)=-\frac{1}{\pi}\int_{-1}^{1}K^{BB}(x,\xi)g^B(\xi)\mathrm{d}\xi+\frac{1}{\pi}\int_{-L_1}^{L_2}K^{DB}(x,\xi)g^D(\xi)\mathrm{d}\xi+2W_{Dt}\frac{\mathrm{d}s_B}{\mathrm{d}x_B} \tag{4}$$

$$g^D(x)=-\frac{1}{\pi}\int_{-1}^{1}K^{BD}(x,\xi)g^B(\xi)\mathrm{d}\xi+\frac{1}{\pi}\int_{-L_1}^{L_2}K^{DD}(x,\xi)g^D(\xi)\mathrm{d}\xi+2W_{Dt}\frac{\mathrm{d}s_D}{\mathrm{d}x_D} \tag{5}$$

where

$$g(x)=\gamma(s)\frac{\mathrm{d}s}{\mathrm{d}x},\ W_{Dt}\frac{\mathrm{d}s}{\mathrm{d}x}=u_{D_1}+u_{D_2}+(v_{D_1}+v_{D_2})\frac{\mathrm{d}r}{\mathrm{d}x}$$

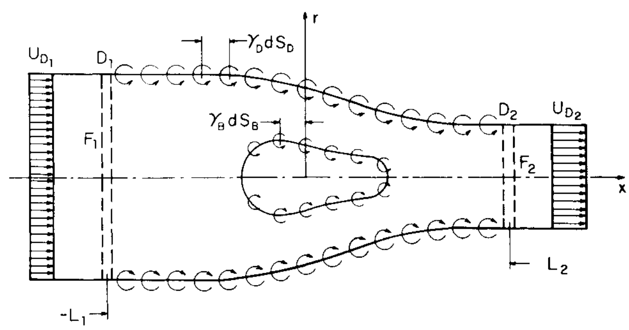

Fig. 1 Definition sketch.

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and the kernels are defined as

$$K(x,\xi)=\frac{1}{r(\xi)}\left[u^*(x,\xi)+v^*(x,\xi)\frac{dr}{dx}\right] \tag{6}$$

and expressed by means of the complete elliptic integrals of the first and second kind $K(k)$ and $E(k)$ of modulus

$$k^2=\frac{4r(x)r(\xi)}{(x-\xi)^2+[r(\xi)+r(x)]^2}$$

$$K(x,\xi)=\frac{1}{\sqrt{(x-\xi)^2+[r(\xi)+r(x)]^2}}\times\left\{\left[1-\frac{r'(x)}{r(x)}(x-\xi)\right][K(k)-E(k)]-\frac{2r(\xi)[r(x)-r(\xi)-r'(x)(x-\xi)]}{(x-\xi)^2+[r(\xi)-r(x)]^2}E(k)\right\} \tag{7}$$

The solution of the coupled integral equations of Fredholm of the second kind gives directly the velocity distribution on the body and duct surfaces. The values for $\gamma(s)$ obtained after integration give the velocity at any point of the flow.

It can be shown that, when the radius of the duct R tends to infinity, $u_{D1}=0.5$, $u_{D2}=0.5$, $v_{D1}=v_{D2}=0$, and the following equation for unbounded fluid is obtained

$$g^B(x)=2-\frac{1}{\pi}\int_{-1}^{1}K^{BB}(x,\xi)g^B(\xi)d\xi \tag{8}$$

Substitution of $\gamma^D=U_\infty+\gamma_D'$ into Eq. (4) yields, in the case of $R=$const., to the equations of Ref. 2.

For the numerical solution of the integral equations, a suitable quadrature formula is used, leading to an algebraic system of equations. The solution of this system gives the velocity on the body and duct wall at the discrete points of the quadrature.

Considering the singular case $\xi\to x$ and removing the logarithmic singularity, we have from Eq. (7)

$$\underset{\xi\to x}{K(x,\xi)}=\frac{1}{2r(x)}\left[\ln\frac{8r(x)}{|x-\xi|[1+(dr/dx)^2]^{1/2}}+\frac{r(x)d^2r/dx^2}{1+(dr/dx)^2}-1\right] \tag{9}$$

This yields the algebraic system

$$g_i\left\{1+\text{sign}(N-i)\left[\frac{1}{\pi}\sum_{j=n}^{m}{}'\lambda_jA_{ij}+\frac{1}{\pi}\lambda_iK_{ii}-T_i\right]\right\}+\frac{1}{\pi}\sum_{j=1}^{N+M}\text{sign}(N-j)\lambda_jK_{ij}g_j=D_i \tag{10}$$

where $i=1,2,3,\ldots,N,N+1,\ldots N+M$

$$n=\begin{cases}1, & i\le N\\ N+1, & i>N\end{cases}\qquad m=\begin{cases}N, & i\le N\\ M, & i>N\end{cases}\qquad \text{sign}(0)=1$$

λ_j = weighting factors of the quadrature formula
N,M = number of control points on body and duct, respectively,

$$D_i=2\left[u_{D_1i}+u_{D_2i}+\frac{dr}{dx}(u_{D_1i}+u_{D_2i})\right]$$

$$A_{ij}=\frac{1}{2r_i}\ln|x_i-x_j|$$

$$T_i=\frac{1}{2\pi r_i}[(x_i+a)\ln(x_i+a)+(b-x_i)\ln(b-x_i)-(a+b)]$$

$$a=\begin{cases}1, & i\le N\\ L_1, & i>N\end{cases}$$

$$b=\begin{cases}1, & i\le N\\ L_2, & i>N\end{cases}$$

It is known[5] that the velocity induced by a vortex cylinder is expressed in terms of the complete elliptic integrals of the first, second, and third kind. Special care is needed for evaluating the term $(r-1/r+1)\Pi(n,\alpha)$, where Π is the complete elliptic integral of the third kind appearing in the expression for the velocity. When $r\to 1$, Π increases and the computational error becomes considerable. To diminish the

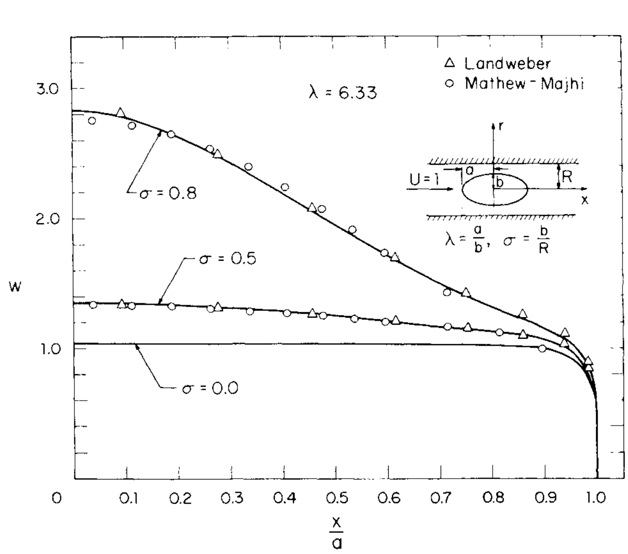

Fig. 2 Velocity distribution on a spheroid for various blockages.

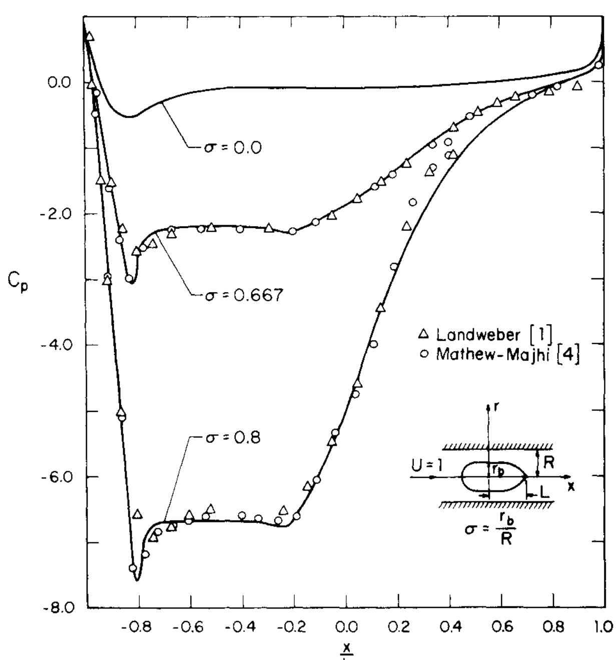

Fig. 3 Pressure coefficient distribution on a body described in Ref. 2 for various blockages.

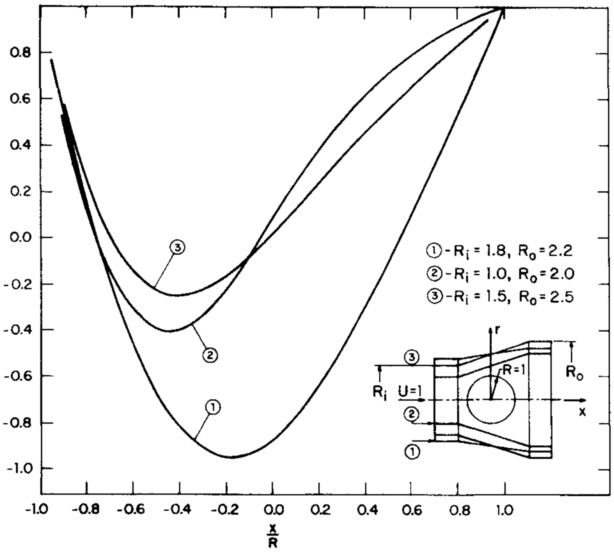

Fig. 4 Pressure coefficient distribution on a sphere placed in a conical duct.

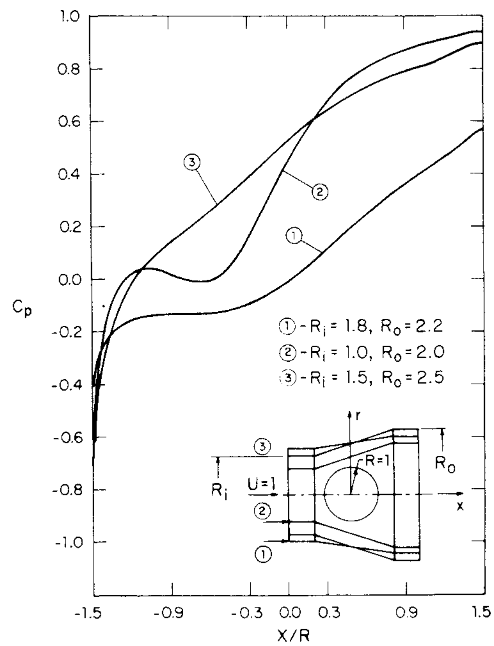

Fig. 5 Pressure coefficient distribution on a conical duct containing a central sphere.

error, the factor $(r-1/r+1)$ is introduced in the expressions for Π, as shown in Ref. 8.

Of interest is the transition between the vortex system on the duct and the source discs because of the singularity of the radial velocity component at the edge of the discs at $x=-L_1$, L_2. The numerical results have shown that it is more convenient to include these points in the vortex system of the duct of strength equal to that of the discs.

In the case of high blockages or bodies like ring foils, there are points situated close to the singularities, causing sharp peaks of the kernel function, and the use of a simple quadrature formula leads to a large error. It is believed that the difficulties in Ref. 2 are due to an insufficient number of points because of the limited capacity of the computer available. The investigations of the present work show that, in principle, increasing the number of points does not guarantee that a better solution will be obtained because of the behavior of the kernel functions. In the present work, a linearized method is used.[7] The velocity due to an axisymmetric vortex sheet is considered at a point M close to the vortex surface. As shown in Ref. 7, the use of the trapezoidal rule leads to a large error for the velocity in a cylindrical duct, for instance. This difficulty can be avoided by evaluating the integral for the induced velocity exactly in the small neighborhood of M, using a linearized form for the strength of the vortex sheet. The accuracy of the solution depends on the two parameters p and q, where p denotes the distance of the point M from the surface, and q is the length of the integration interval. The numerical criterion for the application of the linearized solution derived in Ref. 7 can be used in terms of the blockage and length-diameter ratios.

III. Numerical Results

Numerical results for the velocity distribution on the surface of a spheroid with length-diameter ratio $\lambda=6.33$ are plotted in Fig. 2. The blockage varies from $\sigma=0$ to $\sigma=0.8$. The comparison with programmed solutions of Refs. 1 and 4 shows good agreement. The behavior of the pressure distribution on the surface of a body having jumps of curvature is given on Fig. 3 for different blockages.

Figures 4 and 5 show the pressure distribution on a sphere centrally situated in a conical duct. Let R_i be the inlet radius, and R_0 be the outlet radius. Defining $\sigma=2R_B/(R_i+R_0)$, and h = outlet/inlet area ratio, the three plotted cases are

1) $h=1.49$, $\sigma=0.5$
2) $h=4.0$, $\sigma=0.667$
3) $h=2.778$, $\sigma=0.5$

The pressure on the surfaces is a function of the both parameters. With increasing h, the point of minimum pressure moves forward. For equal σ, the value of the minimum increases with decreasing h. Also of interest is the behavior of the pressure on the duct (Fig. 5). The increase of σ can give rise to a negative pressure gradient which may serve to prevent a boundary-layer separation.

As a concluding remark, it should be mentioned that the present method and computer program can be applied for economically solving engineering axisymmetric flow problems.

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